Vibration Analysis of Arbitrary Quadrilateral Unsymmetrically Laminated Thick Plates

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I. Introduction

CONSIDERABLE work has been devoted to the study of free vibration behavior of fiber-reinforced composite laminated plates.[1] The research was initiated because a fundamental understanding of the vibration behavior is of practical importance in many engineering applications. In this Note, the authors extend their earlier work[2–6] to study the free vibration of thick unsymmetrically laminated quadrangular plates.

The p–Ritz method[2–6] is employed to derive the governing eigenvalue equation of the problem. The first-order shear deformable plate theory proposed by Yang et al.[7] (YNS) is used to account for the effects of the transverse shear deformation. The analysis method is capable of handling unsymmetric composite laminates of different boundary conditions, an arbitrary quadrilateral geometry, and anisotropic material properties. Thus, we believe a numerical tool with such capabilities, as proposed, is of great value for preliminary design of composite structures.

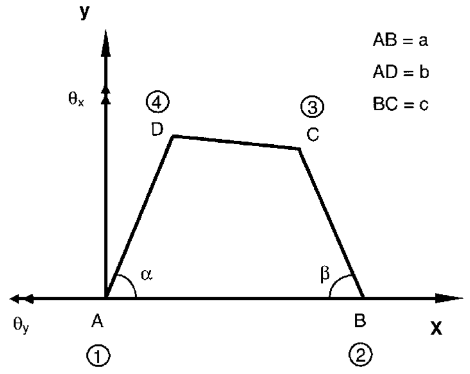
In this Note, convergence characteristics of the p–Ritz method are demonstrated through numerical examples. The accuracy of the results is verified by comparison with the existing literature. Moreover, the ANSYS finite element package is used to analyze the same example problems, and these finite element results can be used to further validate the accuracy of the p–Ritz method.

II. Mathematical Formulations

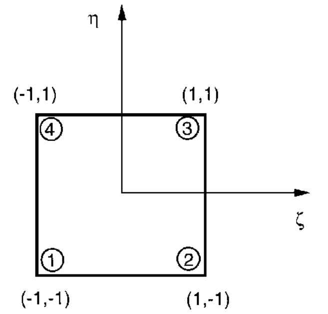
Consider an arbitrary flat quadrilateral plate of uniform thickness h, composed of any number of anisotropic plies oriented alternately at angles θ and $-\theta$. The Cartesian coordinate system x–y located at the middle plane of the plate and the geometry of the plate with side lengths a, b, and c, and two angles α and β define the quadrilateral geometry (Fig. 1a). The material of each ply is assumed to possess a plane of elastic symmetry parallel to the x–y plane.

The plate under consideration is subjected to different combinations of free, simply supported, and clamped boundary conditions. The plate (Fig. 1) is described by a symbol defining the boundary conditions at their four edges, for instance, SCSF means a plate whose edges at AB, BC, CD, and AD are simply supported, clamped, simply supported, and free, respectively. The problem is to determine the natural frequencies of the plate.

The p–Ritz method, which was applied to solve plates in a rectangular domain, has been extended to an arbitrary quadrilateral domain using a geometric mapping technique. For convenience in numerical integration and application of the boundary conditions, the actual quadrilateral domain in the x–y plane has been mapped into a 2×2 basic square domain in the ζ–η plane (Fig. 1b) using the coordinate transformation defined by



a) Actual quadrilateral plate



b) Basic square plate

Fig. 1 Geometry and coordinate systems.

$$x = \sum_{i=1}^{4} N_i x_i \tag{1a}$$

$$y = \sum_{i=1}^{4} N_i y_i \tag{1b}$$

where x_i and y_i are the coordinates of the ith corner of the quadrilateral plate in the x–y plane.

The mapping functions N_i are defined by

$$N_i = \tfrac{1}{4}(1 + \zeta\zeta_i)(1 + \eta\eta_i) \qquad i = 1, 2, 3, 4 \tag{2}$$

where ζ_i and η_i are the coordinates of the ith corner of the basic square domain in the ζ–η plane.

Using the chain rule of differentiation, the first derivatives of any quantity ( ) in the two coordinate systems are related by

$$\begin{Bmatrix} \partial_x(\,) \\ \partial_y(\,) \end{Bmatrix} = \boldsymbol{J}^{-1} \begin{Bmatrix} \partial_\zeta(\,) \\ \partial_\eta(\,) \end{Bmatrix} \tag{3}$$

where

$$\boldsymbol{J} = \begin{bmatrix} \partial_\zeta x & \partial_\zeta y \\ \partial_\eta x & \partial_\eta y \end{bmatrix} \tag{4}$$

in which $\boldsymbol{J}$ denotes the Jacobian matrix of the geometric mapping.

For the laminated plate considered, the displacement and rotation components may be expressed by a set of p–Ritz functions in the ζ–η plane as

$$u(\zeta, \eta) = \sum_{}^{p_1}\sum_{}^{q} a_i \phi_{xi}(\zeta, \eta) = \sum_{i=1}^{m_1} a_i \phi_{xi}(\zeta, \eta) = \boldsymbol{a}^T \phi_x \tag{5}$$

$$v(\zeta, \eta) = \sum_{}^{p_2}\sum_{}^{q} b_i \phi_{yi}(\zeta, \eta) = \sum_{i=1}^{m_2} b_i \phi_{yi}(\zeta, \eta) = \boldsymbol{b}^T \phi_y \tag{6}$$

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$$w(\zeta, \eta) = \sum_{q=0}^{p_3}\sum_{r=0}^{q} c_i \phi_{zi}(\zeta, \eta) = \sum_{i=1}^{m_3} c_i \phi_{zi}(\zeta, \eta) = \boldsymbol{c}^T \boldsymbol{\phi}_z \tag{7}$$

$$\theta_x(\zeta, \eta) = \sum_{q=0}^{p_4}\sum_{r=0}^{q} d_i \psi_{xi}(\zeta, \eta) = \sum_{i=1}^{m_4} d_i \psi_{xi}(\zeta, \eta) = \boldsymbol{d}^T \boldsymbol{\psi}_x \tag{8}$$

$$\theta_y(\zeta, \eta) = \sum_{q=0}^{p_5}\sum_{r=0}^{q} e_i \psi_{yi}(\zeta, \eta) = \sum_{i=1}^{m_5} e_i \psi_{yi}(\zeta, \eta) = \boldsymbol{e}^T \boldsymbol{\psi}_y \tag{9}$$

where p_s ($s = 1, 2, 3, 4, 5$) is the degree of the mathematically complete two-dimensional polynomial space; a_i, b_i, c_i, d_i, and e_i are the unknown coefficients to be varied with the subscript i given by

$$i = \frac{(q + 1)(q + 2)}{2} - (q - r) \tag{10}$$

in which $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, and $\boldsymbol{e}$ are the unknown coefficient vectors having a_i, b_i, c_i, d_i, and e_i as respective elements; m_s ($s = 1, 2, 3, 4, 5$) are, respectively, the dimensions of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, and $\boldsymbol{e}$ given by

$$m_s = \tfrac{1}{2}(p_s + 1)(p_s + 2) \tag{11}$$

$\boldsymbol{\phi}_x$, $\boldsymbol{\phi}_y$, $\boldsymbol{\phi}_z$, $\boldsymbol{\psi}_x$, and $\boldsymbol{\psi}_y$ are the p–Ritz function vectors whose elements are given by

$$\phi_{xi} = f_i \phi_{x1} \tag{12}$$

$$\phi_{yi} = f_i \phi_{y1} \tag{13}$$

$$\phi_{zi} = f_i \phi_{z1} \tag{14}$$

$$\psi_{xi} = f_i \psi_{x1} \tag{15}$$

$$\psi_{yi} = f_i \psi_{y1} \tag{16}$$

where

$$f_i = \zeta^{q-r} \eta^r \tag{17}$$

and ϕ_{x1}, ϕ_{y1}, ϕ_{z1}, ψ_{x1}, and ψ_{y1} are the basic functions corresponding to u, v, w, θ_x, and θ_y, respectively. The basic functions consist of the products of boundary expressions of the laminated plate to ensure the automatic satisfaction of geometric boundary conditions.[4]

Following the p–Ritz procedures,[2–6] the governing eigenvalue equation of the problem has been derived and is given by

$$(\boldsymbol{K} - \Omega^2 \boldsymbol{M})\boldsymbol{q} = 0 \tag{18}$$

Standard eigenvalue solvers may be used to compute the natural frequencies of laminated quadrilateral plates by solving the general eigenvalue problem defined in Eq. (18).

III. Numerical Results and Discussion

A high-modulus graphite/epoxy has been used to study the vibration behavior of the unsymmetrically laminated composite plates. Each ply is a unidirectional fiber-reinforced composite possessing the dimensionless material properties of $E_1/E_2 = 40$, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, and $\nu_{12} = 0.25$.

Table 1 presents the calculated natural frequency parameters for a moderately thick isotropic plate with a relative thickness ratio $t/a = 0.10$. These results are compared with the values from the Mindlin plate theory.[8,9] For numeric computation, the degree of polynomials p_i increases from 4 to 10, which is equivalent to a variation in determinant size from 75×75 to 330×330. Clearly, the eigenvalues of lower modes converge relatively faster than the higher modes, and $p_i = 10$ is needed to give convergent eigenvalues. It is evident that the present results are in close agreement with the Mindlin solutions.[8,9]

Table 1 Comparison of frequency parameter $\lambda = \omega a^2 \sqrt{[\rho/(Eh^2)]}$ of a square simply supported isotropic plate ($\nu = 0.3$, $t/a = 0.1$)

Sources	Mode sequence number 1	2	3	4	5
$p_s = 4$	5.770	13.804	27.561	33.205	46.581
$p_s = 6$	5.769	13.764	26.040	32.689	43.567
$p_s = 8$	5.769	13.764	25.738	32.295	43.169
$p_s = 9$	5.769	13.764	25.734	32.294	42.421
$p_s = 10$	5.769	13.764	25.734	32.284	42.420
Mindlin's solution[8]	5.77	13.7	25.7	32.2	42.3
Reddy's finite element solution[9]	5.793	14.081	27.545	35.050	49.693

In Table 2, the fundamental frequency parameters for a simply supported three-ply orthotropic square laminated plate are given together with the values of the shear deformation theory and classical lamination theory.[10] A convergence study has again been carried out by varying the number of degree of polynomials p_i for the laminated plate with various degree of orthotropy E_1/E_2. It is obvious that a degree p_i of less than 4 (the determinant size is 75×75) is more than enough to furnish convergent results inasmuch as only the fundamental frequency parameters are of interest in this comparison. The results obtained from the present analysis and the shear deformation theory of Whitney and Pagano[10] are again in close agreement.

Table 2 Comparison of nondimensional fundamental frequency $\omega h \sqrt{[\rho_2/(E_x)_2]}$ of a simply supported three-ply orthotropic square plate ($h_1 : h_2 : h_3 = 0.1:0.8:1$, $h/a = 0.1$)

Sources	$(E_x)_1/(E_x)_2$ 1	2	3	4	5
$p_s = 2$	0.0501	0.0605	0.0839	0.1122	0.1345
$p_s = 4$	0.0474	0.0573	0.0796	0.1065	0.1278
$p_s = 6$	0.0474	0.0573	0.0796	0.1065	0.1278
Whitney and Pagano[10]	0.0477	0.0578	0.0804	0.1077	0.1293
Whitney and Pagano[10]	0.0470	0.0567	0.0785	0.1048	0.1256
Whitney and Pagano[10]	0.0474	0.0573	0.0796	0.1065	0.1278

Table 3 Nondimensional frequency parameter $\lambda = \omega a^2 \sqrt{[\rho/(E_2 h^2)]}$ of a fully clamped four-ply quadrilateral laminated plate ($\theta/-\theta/\theta/-\theta$)

θ, deg	h/a	Sources	Mode sequence number 1	2	3	4	5
30	0.10	Authors	34.267	51.511	58.660	72.589	76.676
		ANSYS	33.721	48.286	59.739	——	76.485
	0.20	Authors	19.516	29.426	31.744	41.000	41.245
		ANSYS	19.315	27.842	32.743	37.951	41.716
45	0.10	Authors	35.482	56.347	57.741	77.789	81.190
		ANSYS	34.355	52.332	58.408	73.114	77.834
	0.20	Authors	19.895	30.776	31.456	41.394	43.274
		ANSYS	19.541	29.128	32.227	39.890	42.298
60	0.10	Authors	34.222	51.952	58.440	72.492	76.518
		ANSYS	32.571	50.755	55.547	70.142	77.137
	0.20	Authors	19.483	29.581	31.660	40.285	41.744
		ANSYS	18.979	28.971	31.056	39.244	42.590

Table 3 shows a comparison of nondimensional frequency parameters for a fully clamped four-ply unsymmetrically laminated quadrilateral plate with stacking sequence ($\theta/-\theta/\theta/-\theta$) of relative thickness ratios $h/a = 0.10$ and 0.20. The plate geometry is defined by parameters given as follows: $b/a = 0.9$, $c/a = 0.7$, $\alpha = 65$ deg, and $\beta = 75$ deg (as shown in Fig. 1). The ANSYS version 5.2 finite element package was also used to model the problem using the SOLID46 element with a $20 \times 20 \times 10$ mesh. In general, the authors' results are in good agreement with the ANSYS finite element solutions.

IV. Concluding Remarks

This Note considers the vibration analysis of arbitrary quadrilateral unsymmetrically laminated composite plates using the p–Ritz method. The transverse shear deformation effect was incorporated in the mathematical model via the YNS first-order shear deformation theory.[7] The accuracy of the method was established through convergence and comparison studies. Because of a lack of published data for arbitrary quadrilateral laminated plates, the ANSYS finite element package was used to generate solutions for the purpose of comparison. In general, close agreements are obtained for all cases, thus verifying the accuracy of the present method.

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